

Optimality Conditions for Bilevel Programming Problem in Asplund Spaces

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Abstract

Here, necessary optimal condition for Optimistic Bilevel programming problem is obtained in Asplund spaces. Also we have got necessary optimal conditions in finite dimensional spaces, by assuming differentiability on the given functions.

Key Word Asplund Space, Optimization, nonsmooth calculus

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1 Introduction

Bilevel programming problem was first formulated by H.V. Stackelberg [10] in his monograph on market economy on 1934. A special type of bilevel programming problems named as Stackelberg games were considered over years within economic game theory. For the optimization community these problems were first introduced only in seventies of the last century. These types of problems are faced in engineering, economics and in other disciplines also. From the mathematical point of view, bilevel programming problems are complicated and NP-hard problems.

Bilevel programming problems are special types of optimization problems where another parametric optimization problem is contained as a part of the constraints of the first problems. Also we would like to mention that bilevel programming problem in infinite dimensional spaces are not new. For infinite dimensional bilevel programming problem one may refer [7] and [1]. Here our main aim is to study bilevel programming problem in Asplund space. In the language of optimization, bilevel programming problem (BP) can be framed as a two level problem as follows:

$$\min_x F(x, y) \quad \text{subject to} \quad x \in \Omega \quad \text{and} \quad y \in S(x),$$

where X and Y are asplund spaces, $F : X \times Y \rightarrow \mathbb{R}$, $\Omega \subset X$, and $S : X \rightrightarrows Y$ is the solution set mapping to the lower-level problem:

$$\min_y f(x, y) \quad \text{subject to} \quad y \in K(x),$$

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where $f : X \times Y \rightarrow \mathbb{R}$ and $K(x)$ is closed for all x . The first problem is called as upper level problem and second one is called as lower level problem. The basic idea of bilevel problem is that the upper-level decision maker, or the leader chooses a decision vector x and passes it into the lower-level decision maker, or the follower, who then, based on the leader's choice x , minimizes his objective function and returns the solution y to the leader who then uses it to minimize his objective function.

If for each x the lower level problem has a unique solution, then problem(BP) is well defined. However if there are many solution to the lower level problem for a given x , then the objective function of upper level problem becomes a set-valued map. In order to overcome this difficulty, two different solutions concept known as optimistic solution and pessimistic solution are defined in the literature. For the optimistic case, one first defines the function

$$\phi_0(x) = \inf_y \{F(x, y) : y \in S(x)\}.$$

Then the optimistic problem is:

$$\min \phi_0(x) \quad \text{subject to} \quad x \in X. \quad (1.1)$$

A pair of points (\bar{x}, \bar{y}) is said to be optimistic solution to the bilevel problem (BP) if $\phi_0(\bar{x}) = F(\bar{x}, \bar{y})$ and \bar{x} is the optimal solution to 1.1. On the other hand for the pessimistic case one defines function

$$\phi_1(x) = \sup_y \{F(x, y) : y \in S(x)\}.$$

Then the pessimistic problem is:

$$\min \phi_1(x) \quad \text{subject to} \quad x \in X. \quad (1.2)$$

A pair of points (\bar{x}, \bar{y}) is said to be pessimistic solution to the bilevel problem (BP) if $\phi_1(\bar{x}) = F(\bar{x}, \bar{y})$ and \bar{x} is the optimal solution to 1.2. In this chapter we are considering only optimistic bilevel programming. Optimistic bilevel programming problems are very useful in practical application like supply chain management problems. As it is explained in Dutta and Dempe [2], to study above type of optimistic bilevel problem(BP), it is useful to concentrate on the following problem:

$$\min_{x,y} F(x, y) \quad \text{subject to} \quad x \in \Omega, (x, y) \in \text{gph}S,$$

where $S(x)$ is the solution set. If we assume $f(x, \cdot)$ is fréchet differentiable with respect to y and $K(x)$ is closed convex in bilevel programming problem (BP) then $S(x)$ can be written as

$$S(x) = \{y \in Y : 0 \in \nabla_y f(x, y) + N_{K(x)}(y)\}.$$

Setting $N_K(x, y) = N_{K(x)}(y)$ if $y \in K(x)$ and $N_K(x, y) = \emptyset$ otherwise, we rewrite $S(x)$ as

$$S(x) = \{y \in Y : 0 \in \nabla_y f(x, y) + N_K(x, y)\}.$$

So now we will write formally the problem as follows:

$$\begin{aligned} \min_{x, y} F(x, y) \quad \text{subject to} \quad & x \in \Omega, (x, y) \in \text{gph} S, \\ & S(x) = \{y \in Y : 0 \in \nabla_y f(x, y) + N_K(x, y)\}, \end{aligned}$$

which is equivalently

$$\min_{x, y} F(x, y) \quad \text{subject to} \quad 0 \in \nabla_y f(\bar{x}, \bar{y}) + N_K(x, y), \quad x \in \Omega. \quad (1.3)$$

2 Preliminaries and Notations

Asplund spaces are a subclass of Banach spaces in which every continuous convex function is generically Frechet differentiable, i.e., differentiable over a dense G_δ set. The class of Asplund spaces contains all Hilbert spaces and reflexive Banach spaces. There are many interesting characterizations of Asplund spaces. See for example Phelps [9]. As usual for any Banach space X we shall denote its dual by X^* . Further by the symbol $x_k \xrightarrow{w^*} x$ we mean x_k converges to x with respect to weak* topology. We denote by $\langle \cdot, \cdot \rangle$ the usual duality pairing between X and X^* . For any function $F : X \rightarrow Y$ and $y^* \in Y^*$, we evaluate the function $\langle y^*, F \rangle : X \rightarrow \mathbb{R}$ as $\langle y^*, F \rangle(x) = \langle y^*, F(x) \rangle$. Also graph of a function (set-valued function) F is denoted as $\text{gph} F$.

Further let us note that set-valued maps or multi-valued functions play a major role in our analysis. We shall begin by presenting the definition of the Fréchet normal cone. This notion can be viewed as a geometric building block to study nonsmooth analysis in Asplund spaces.

Definition 1 *Let Ω be a non-empty subset of the Asplund space X and let $\bar{x} \in \Omega$. Then the Frechet normal cone to Ω at \bar{x} which is denoted as $\hat{N}(\bar{x}, \Omega)$ and is given as*

$$\hat{N}(\bar{x}, \Omega) = \{x \in X^* \mid \limsup_{x \in \Omega, x \rightarrow \bar{x}} \frac{\langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \leq 0\}.$$

Though the definition of the Frechet normal cone appears simple there is a major drawback in the construction. The Frechet normal cone can become just the trivial cone containing only the zero vector. For that reason, a new cone named as the *basic normal cone* or the *Mordukhovich normal cone* is developed and this is what we present next.

Definition 2 *Let X be an Asplund space and $\Omega \subset X$ be a non-empty closed set and let $x \in S$. The basic normal cone or the Mordukhovich normal cone to Ω at \bar{x} is*

$$N(\bar{x}, \Omega) := \{x^* \in X^* \mid \exists x_n \xrightarrow{\Omega} \bar{x}, x_n^* \xrightarrow{w^*} x^*, x_n^* \in \hat{N}(x_n, \Omega)\}$$

This can be more compactly written as

$$N(\bar{x}, \Omega) = \limsup_{x \rightarrow \bar{x}} \hat{N}(x, \Omega).$$

The notion of the basic normal cone was introduced by Mordukhovich [3] motivated by problems in optimal control. The basic normal cone is a robust object in the finite dimensional setting in the sense that its graph is closed when viewed as a set-valued map. Such a property however does not hold true in Asplund spaces and this makes the analysis in Asplund spaces more involved. Further in contrast to the Frechet normal cone which is a convex set the basic normal cone need not be a convex set. For a detailed theory of basic normal cones in Asplund spaces see for example Mordukhovich [5]. Let us turn our attention now to the issues of differentiability in Asplund Spaces. In infinite dimensional space, a function is differentiable in Fréchet sense means that a function $f : X \rightarrow Y$ is Fréchet differentiable at \bar{x} if there is a linear operator $\nabla f(\bar{x}) : X \rightarrow Y$, called the Fréchet derivative of f at \bar{x} , such that

$$\lim_{x \rightarrow \bar{x}} \frac{f(x) - f(\bar{x}) - \nabla f(\bar{x})(x - \bar{x})}{\|x - \bar{x}\|} = 0.$$

But for application purpose, we need a stronger differentiability property which is called as strict differentiability property, i.e., $f : X \rightarrow Y$ is strict differentiable at \bar{x} if

$$\lim_{x \rightarrow \bar{x}, u \rightarrow \bar{x}} \frac{f(x) - f(u) - \nabla f(\bar{x})(x - u)}{\|x - u\|} = 0.$$

The set of strictly differentiable mappings include the class of continuously differentiable mappings. Note that every mapping f strictly differentiable

at \bar{x} is Lipschitzian around \bar{x} , i.e. there is a neighborhood U of \bar{x} and a constant $l \geq 0$ such that

$$\|f(x) - f(u)\| \leq l \|x - u\| \quad \text{for all } x, u \in U.$$

But for application point of view, we require stronger Lipschitzian property which is called as strictly Lipschitzian property and is defined below.

Definition 3 Let $f : X \rightarrow Y$ be a single-valued mapping between Banach spaces. Assume that f is Lipschitzian at \bar{x} . Then f is called as strictly Lipschitzian at \bar{x} if there is a neighborhood V of the origin in X such that the sequence

$$y_k = \frac{f(x_k + t_k v) - f(x_k)}{t_k}, \quad k \in \mathbb{N},$$

contains a norm convergent subsequence whenever $v \in V, x_k \rightarrow \bar{x}$ and $t_k \downarrow 0$.

Remark 1 When Y is finite dimensional space, strictly Lipschitzian is same as a locally Lipschitz function. Note that the set of strictly Lipschitzian function at a point is closed with respect to composition and also form a linear space. For more property of strictly Lipschitzian function see [5].

Also we will require Lipschitz-like properties and calmness properties for set valued mapping that are defined as follows:

Definition 4 Let $F : X \rightrightarrows Y$ be a set-valued mapping with $\text{dom } F \neq \emptyset$. Given $(\bar{x}, \bar{y}) \in \text{gph } F$, we say that F is locally Lipschitz-like around (\bar{x}, \bar{y}) with modulus $l \geq 0$ if there are neighborhood U of \bar{x} and V of \bar{y} such that

$$F(x) \cap V \subset F(u) + l \|x - u\| \mathbb{B} \quad \text{for all } x, u \in U,$$

holds.

Definition 5 Let $F : X \rightrightarrows Y$ be set-valued mapping between Banach spaces. Given $(\bar{x}, \bar{y}) \in \text{gph } F$, we say that F is calm at (\bar{x}, \bar{y}) with modulus $l \geq 0$ if there are neighborhood U of \bar{x} and V of \bar{y} such that

$$F(x) \cap V \subset F(\bar{x}) + l \|x - \bar{x}\| \mathbb{B} \quad \text{for all } x \in U,$$

holds.

It is easy to observe that calmness property is implied by Lipschitz like property. In definition of Lipschitz like property, \bar{x} is replaced by $u \in U$ that varies around \bar{x} together with x . Further if F is single-valued then the Lipschitz-like property reduces to the usual notion of locally Lipschitz function.

To handel non-smoothness we will use here *basic subdifferential* or the *Mordukhovich subdifferential* which we present next.

Definition 6 Consider a function $\phi : X \rightarrow \bar{\mathbb{R}}$ and a point $\bar{x} \in X$ with $|\phi(\bar{x})| < \infty$. The set

$$\partial\phi(\bar{x}) = \{x^* \in X^* | (x^*, -1) \in N((\bar{x}, \phi(\bar{x})), \text{epi}\phi)\},$$

is defined as the *basic subdifferential* of ϕ at \bar{x} and its elements are *basic subgradients* at this point. If $\phi(\bar{x}) = +\infty$ then it is customary to set $\partial\phi(\bar{x}) = \emptyset$.

The basic subdifferential is non-empty when ϕ is locally Lipschitz and in the finite dimensional setting is a compact set. However the important point to note is that the basic subdifferential need not be a convex set and may not even be closed in Hilbert spaces. For details on the basic subdifferential see for example Mordukhovich [5]. The basic subdifferential was first introduced by Mordukhovich [4] in 1980 and is currently one of the central tools in nonsmooth optimization.

Further there is a notion of *singular subdifferential* which in some sense measures how much a function has deviated from the Lipschitz property at a given point.

Definition 7 Consider a function $\phi : X \rightarrow \bar{\mathbb{R}}$ and a point $\bar{x} \in X$ with $|\phi(\bar{x})| < \infty$. The set

$$\partial^\infty\phi(\bar{x}) = \{x^* \in X^* | (x^*, 0) \in N((\bar{x}, \phi(\bar{x})), \text{epi}\phi)\},$$

is defined as the *singular subdifferential* of ϕ at \bar{x} .

It is interesting to note that if f is Lipschitz at \bar{x} then $\partial^\infty f(\bar{x}) = \{0\}$. *coderivative* plays fundamental role in defining derivative like object in a set-valued map. We present three different notions of the coderivative from Mordukovich [5].

Definition 8 Let $F : X \rightrightarrows Y$ be a set-valued map. Then the *Frechet coderivative* of F at $(\bar{x}, \bar{y}) \in \text{gph } F$ is defined as the multifunction $\hat{D}^*F(\bar{x}, \bar{y}) : Y^* \rightrightarrows X^*$ such that

$$\hat{D}^*F(\bar{x}, \bar{y})(y^*) = \{x^* \in X^* | (x^*, -y^*) \in \hat{N}((\bar{x}, \bar{y}), \text{gph } F)\}.$$

Definition 9 Let $F : X \rightrightarrows Y$ be a set-valued map. Then the mixed coderivative of F at the $(\bar{x}, \bar{y}) \in \text{gph } F$ is a set-valued map $D_M^* : Y^* \rightrightarrows X^*$ given as

$$D_M^*F(\bar{x}, \bar{y})(\bar{y}^*) = \limsup_{(x, y) \rightarrow (\bar{x}, \bar{y}), y^* \rightarrow \bar{y}^*} \hat{D}^*F(x, y)(y^*).$$

This means that the mixed coderivative is a collection of $\bar{x}^* \in X^*$ such that there are sequences $(x_k, y_k, y_k^*) \rightarrow (\bar{x}, \bar{y}, \bar{y}^*)$ and $x_k^* \xrightarrow{w^*} \bar{x}^*$ with $(x_k, y_k) \in \text{gph } F$ and $x_k^* \in \hat{D}^*F(x, y)(y_k^*)$.

Definition 10 Let $F : X \rightrightarrows Y$ be a set-valued map. Then the normal coderivative of F at the $(\bar{x}, \bar{y}) \in \text{gph } F$ is a set-valued map $D_N^* : Y^* \rightrightarrows X^*$ given as

$$D_N^*F(\bar{x}, \bar{y})(\bar{y}^*) = \limsup_{(x, y) \rightarrow (\bar{x}, \bar{y}), y^* \xrightarrow{w^*} \bar{y}^*} \hat{D}^*F(x, y)(y^*).$$

This means that the mixed coderivative is a collection of $\bar{x}^* \in X^*$ such that there are sequences $(x_k, y_k) \rightarrow (\bar{x}, \bar{y})$ and $(x_k^*, y_k^*) \xrightarrow{w^*} (\bar{x}^*, \bar{y}^*)$ with $(x_k, y_k) \in \text{gph } F$ and $x_k^* \in \hat{D}^*F(x, y)(y_k^*)$. The normal coderivative can also be equivalently represented as

$$D_N^*F(x, y)(y^*) := \{x^* \in X^* \mid (x^*, -y^*) \in N(\text{gph } F, (x, y))\}.$$

When $F : X \rightarrow Y$ is a single-valued mapping then we denote the mixed and normal coderivatives as $D_M^*F(\bar{x})(y^*)$ and $D_N^*F(\bar{x})(y^*)$ respectively. Observe that the major difference in the definition of the mixed coderivative and normal coderivative is in the manner in which y_k^* converges to \bar{y}^* . In case of the mixed derivative the convergence is in the norm topology while in case of the normal coderivative the convergence is in the *weak*^{*}-star topology. It is important to note that if Y is finite dimensional spaces both the notions of coderivative coincide. Moreover all the three coderivatives mentioned above are positively homogenous set-valued map. A set-valued map $F : X \rightrightarrows Y$ is said to be positively homogeneous if $\alpha F(x) \subset F(\alpha x)$, with $\alpha > 0$. Further it is clear that for any $y^* \in Y^*$ one has

$$\hat{D}F(\bar{x}, \bar{y})(y^*) \subset D_M^*F(\bar{x}, \bar{y})(y^*) \subset D_N^*F(\bar{x}, \bar{y})(y^*). \quad (2.1)$$

Further if $f : X \rightarrow Y$ strictly differentiable at \bar{x} then one has from Theorem 1.38 in Mordukhovich [5] the following fact,

$$D_M^*f(\bar{x})(y^*) = D_N^*f(\bar{x})(y^*) = \{\nabla f(\bar{x})^* y^*\}, \quad \forall y^* \in Y^*, \quad (2.2)$$

where $\nabla f(\bar{x})^*$ denotes the adjoint operator associated with $\nabla f(\bar{x})$. Note that if $f : X \rightarrow Y$ is single-valued and Lipschitz continuous around \bar{x} , then $D_M^* f(\bar{x})(y^*) = \partial \langle y^*, f \rangle(\bar{x})$ for all $y^* \in Y^*$. If f is strictly Lipschitz continuous around \bar{x} , then $D_N^* f(\bar{x})(y^*) = \partial \langle y^*, f \rangle(\bar{x})$ for all $y^* \in Y^*$. There are many properties which are automatically satisfied in finite dimensional spaces. But in infinite dimensional space, those properties require specific attention. For our requirements the notion of *sequential normal compactness*(SNC) and *partial sequential normal compactness*(PSNC) is defined below. We would also like to note that these properties are automatically satisfied in finite dimensions. We shall begin with the definition of the sequential normal compactness and partial sequential normal compactness of the closed set $\Omega \subset X \times Y$.

Definition 11 Let $\Omega \subset X \times Y$ where X and Y are Asplund spaces. The set Ω is said to be *sequentially normally compact* (SNC) at $(\bar{x}, \bar{y}) \in \Omega$ if for any sequence $(x_k, y_k) \xrightarrow{\Omega} (\bar{x}, \bar{y})$ and

$$(x_k^*, y_k^*) \in \hat{N}((x_k, y_k), \Omega) \quad (2.3)$$

and one has the implication $(x_k^*, y_k^*) \xrightarrow{w^*} 0$ implies that $\|x_k^*, y_k^*\| \rightarrow 0$ as $k \rightarrow \infty$. Further Ω is said to be *partially sequentially normally compact* (PSNC) if for any sequence $(x_k, y_k) \xrightarrow{\Omega} (\bar{x}, \bar{y})$ satisfying (2.3) one has the implication $x_k^* \xrightarrow{w^*} 0, \|y_k^*\| \rightarrow 0$ implies that $\|x_k^*\| \rightarrow 0$, as $k \rightarrow \infty$.

We shall now turn our attention to the SNC and PSNC property of a multifunction $F : X \rightrightarrows Y$, where X and Y are Asplund spaces. The multifunction F is said to have the SNC or PSNC property if its graph has the corresponding properties. However we provide below the definition for completeness.

Definition 12 Let $F : X \rightrightarrows Y$ with $(\bar{x}, \bar{y}) \in \text{gph} F$. Then F is said to be *Sequentially Normally Compact*(SNC) at (\bar{x}, \bar{y}) if for any sequence

$$(x_k, y_k, x_k^*, y_k^*) \in (\text{gph} F) \times X^* \times Y^*$$

satisfying $(x_k, y_k) \rightarrow (\bar{x}, \bar{y})$, $x_k^* \in \hat{D}^* F(x_k, y_k)(y_k^*)$ and $(x_k^*, y_k^*) \xrightarrow{w^*} (0, 0)$ one has $\|(x_k^*, y_k^*)\| \rightarrow 0$ as $k \rightarrow \infty$.

Definition 13 Let $F : X \rightrightarrows Y$ with $(\bar{x}, \bar{y}) \in \text{gph} F$. Then F is said to be *Partially Sequentially Normally Compact*(PSNC) at (\bar{x}, \bar{y}) if for any sequence

$$(x_k, y_k, x_k^*, y_k^*) \in (\text{gph} F) \times X^* \times Y^*$$

satisfying $(x_k, y_k) \rightarrow (\bar{x}, \bar{y})$, $x_k^* \in \hat{D}^*F(x_k, y_k)(y_k^*)$, $x_k^* \xrightarrow{w^*} 0$ and $\|y_k^*\| \rightarrow 0$ one has $\|x_k^*\| \rightarrow 0$ as $k \rightarrow \infty$.

When F is single-valued function we will simply omit \bar{y} . Note from the definition that PSNC is implied by SNC. If f is a single-valued function and Lipschitz continuous around x , then it is PSNC at $(\bar{x}, f(\bar{x}))$.

3 Necessary Optimality Conditions

In this section, we will derive necessary optimality condition for above Bilevel Problem (BP) by using described nonsmooth tools and the results obtained in [8]. First consider the following MPEC problem (P).

$$\min f(x, y) \quad \text{subject to} \quad 0 \in F(x, y) + Q(x, y), \quad (x, y) \in \Omega,$$

where $F : X \times Y \rightarrow W$, $Q : X \times Y \rightrightarrows W$, $f : X \times Y \rightarrow \mathbb{R}$, $\Omega \subset X \times Y$ and X, Y and W are Asplund spaces.

Theorem 1 *Let (\bar{x}, \bar{y}) be a local optimal solution to the problem(P1), where F_1 and Q are mapping between Asplund spaces. Assume that f is locally Lipschitz continuous around (\bar{x}, \bar{y}) , that the sets Ω and $\text{gph } Q$ are closed sets, and that Q is SNC at $(\bar{x}, \bar{y}, \bar{z})$, where $\bar{z} = -F(\bar{x}, \bar{y})$. Also assume that F is strictly Lipschitz continuous at (\bar{x}, \bar{y}) and that relations $(-x^*, -y^*, -z^*) \in N_{\text{gph}Q}(\bar{x}, \bar{y}, \bar{z})$ and*

$$(x^*, y^*) \in \partial \langle z^*, F \rangle(\bar{x}, \bar{y}) + N((\bar{x}, \bar{y}), \Omega)$$

holds only for $x^ = y^* = z^* = 0$. Then there is $(-x^*, -y^*, -z^*) \in N_{\text{gph}Q}(\bar{x}, \bar{y}, \bar{z})$ such that the necessary optimality condition,*

$$0 \in \partial f(\bar{x}, \bar{y}) - (x^*, y^*) + \partial \langle z^*, F \rangle(\bar{x}, \bar{y}) + N((\bar{x}, \bar{y}), \Omega), \quad (3.1)$$

is satisfied.

Proof By taking $F_2(x, y) = (x, y)$ in Corollary 19 [8] or referring Corollary 5.50 of [6], we will get the result. \square

In the following theorem we will prove necessary optimal condition for bilevel programming problem (BP).

Theorem 2 Let (\bar{x}, \bar{y}) be local solution to the problem(BPO) and assume $F : X \times Y \rightarrow \mathbb{R}$ is lipschitz continuous at (\bar{x}, \bar{y}) and $\nabla_y f : X \times Y \rightarrow W$ is strictly Lipschitz continuous at (\bar{x}, \bar{y}) . Assume $\text{gph}N_K$ and Ω are closed sets, N_K is SNC at $(\bar{x}, \bar{y}, \bar{z})$, where $\bar{z} = -\nabla_y f(\bar{x}, \bar{y})$, and the relation, $(-x^*, -y^*, -z^*) \in N_{\text{gph}N_k}(\bar{x}, \bar{y}, \bar{z})$ and

$$(x^*, y^*) \in \partial \langle z^*, \nabla_y f \rangle (\bar{x}, \bar{y}) + N(\bar{x}, \Omega) \times \{0\}$$

holds only for $x^* = y^* = z^* = 0$. Then there is $(-x^*, -y^*, -z^*) \in N_{\text{gph}N_k}(\bar{x}, \bar{y}, \bar{z})$ such that the necessary optimality condition,

$$0 \in \partial F(\bar{x}, \bar{y}) - (x^*, y^*) + \partial \langle z^*, \nabla_y f \rangle (\bar{x}, \bar{y}) + N(\bar{x}, \Omega) \times \{0\},$$

is satisfied.

Proof: From equation (1.3) and applying Theorem (1) we get the relation $(-x^*, -y^*, -z^*) \in N_{\text{gph}N_k}(\bar{x}, \bar{y}, \bar{z})$ and

$$\begin{aligned} (x^*, y^*) &\in \partial \langle z^*, \nabla_y f \rangle (\bar{x}, \bar{y}) + N((\bar{x}, \bar{y}), (\Omega \times Y)) \\ (x^*, y^*) &\in \partial \langle z^*, \nabla_y f \rangle (\bar{x}, \bar{y}) + N(\bar{x}, \Omega) \times N(\bar{y}, Y) \end{aligned}$$

holds only for $x^* = y^* = z^* = 0$. Since \bar{y} is an interior point of Y , above equation reduces to

$$0 \in \partial F(\bar{x}, \bar{y}) - (x^*, y^*) + \partial \langle z^*, \nabla_y f \rangle (\bar{x}, \bar{y}) + N(\bar{x}, \Omega) \times \{0\}.$$

Then there is $(-x^*, -y^*, -z^*) \in N_{\text{gph}N_k}(\bar{x}, \bar{y}, \bar{z})$ such that the necessary optimality condition,

$$0 \in \partial F(\bar{x}, \bar{y}) - (x^*, y^*) + \partial \langle z^*, \nabla_y f \rangle (\bar{x}, \bar{y}) + N(\bar{x}, \Omega) \times \{0\},$$

is satisfied. □

Let us assume that $X = \mathbb{R}^n$, $Y = \mathbb{R}^m$, f is twice differentiable and that F is differentiable. Also assume $\nabla F = (F_x, F_y) = (\xi_1, \xi_2)$. It should be noted that in finite dimensional spaces SNC property is not required.

Theorem 3 Let (\bar{x}, \bar{y}) be local solution to the problem(BPO). Assume that $\text{gph}N_K$ and Ω are closed sets, and the relation, $(-\alpha, -\beta, -\gamma) \in N_{\text{gph}N_k}(\bar{x}, \bar{y}, \bar{z})$, where $\bar{z} = -\nabla_y f(\bar{x}, \bar{y})$ and

$$\begin{aligned} (\alpha - \gamma_1 \nabla_{yx} f) &\in N(\bar{x}, \Omega), \\ \beta &= \gamma_2 \nabla_{yy} f, \end{aligned}$$

where $\gamma = (\gamma_1, \gamma_2)$, holds only for $\alpha = \beta = \gamma = 0$. Then there is $(-\alpha, -\beta, -\gamma) \in N_{gphN_k}(\bar{x}, \bar{y}, \bar{z})$ such that the necessary optimality condition,

$$\begin{aligned}\xi_1 - \alpha + \gamma_1 \nabla_{yx} f + \eta_1 &= 0 \\ \xi_2 + \gamma_2 \nabla_{yy} f &= \beta\end{aligned}$$

is satisfied.

Proof

By applying 2, we get that there exists $(-\alpha, -\beta, -\gamma) \in N_{gphN_k}(\bar{x}, \bar{y}, \bar{z})$ and the relation

$$(\alpha, \beta) - (\gamma_1 \nabla_{yx} f, \gamma_2 \nabla_{yy} f) \in N(\bar{x}, \Omega) \times \{0\}, \quad (3.2)$$

where $\gamma = (\gamma_1, \gamma_2)$. which reduces to

$$\begin{aligned}(\alpha - \gamma_1 \nabla_{yx} f) &\in N(\bar{x}, \Omega) \\ \beta &= \gamma_2 \nabla_{yy} f\end{aligned}$$

holds only for $\alpha = \beta = \gamma = 0$. Then there is $(-\alpha, -\beta, -\gamma) \in N_{gphN_k}(\bar{x}, \bar{y}, \bar{z})$ such that the necessary optimality condition,

$$0 \in (\xi_1, \xi_2) - (\alpha, \beta) + (\gamma_1 \nabla_{yx} f, \gamma_2 \nabla_{yy} f) + (\eta_1, 0)$$

is satisfied. It shows that

$$\begin{aligned}\xi_1 - \alpha + \gamma_1 \nabla_{yx} f + \eta_1 &= 0 \\ \xi_2 + \gamma_2 \nabla_{yy} f &= \beta\end{aligned}$$

□

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